Sensors for Attitude Estimation

Matthew Silic
October 18, 2019

University of Florida
1. Sensor Installation Considerations

2. Stochastic Sensor Models

3. Bias Characterization using Allan Deviation Plots

4. Sensor Calibration for Tri-Axial Field Sensors

5. Physical Models for Common MEMs Sensors
Almost all environmental measurable parameters are in analog form.

The analog signal must be converted into digital data in order to be used by digital processors.

An analog-to-digital converter (ADC) converts an analog signal to a binary number.

ADC resolution = number of bits in the output number, $n$.

The “least significant bit” (LSB) voltage is equal to $V_{\text{ref}}/2^n$.

Because the ADC resolves the analog signal into discrete values, there is a round-off error. The maximum “quantization” error is equal to the LSB voltage.

**Figure 1:** Output of an ideal 3 bit ADC.
Aliasing occurs if you under-sample a sinusoidal signal. The signal that is recreated from the samples has a lower frequency than the actual signal. This low-frequency signal is called an alias. To accurately recreate the signal, the sampling frequency must be at least twice the signal frequency.

Figure 2: If under-sampled, a continuous signal (solid line) will appear as a low frequency signal (dashed line).
Aliasing

- In the frequency domain, the alias and the original signal are mirror images of each other; the Nyquist frequency (half the sampling frequency) is the axis of symmetry.
- Algebraically, this is expressed as
  
  \[ f_a = f \mod f_s, \]

  where \( f_a \) is the frequency of the alias, \( f \) is the true frequency and \( f_s \) is the sampling frequency.

**Figure 3:** The alias and the original signal are mirror images of each other in the frequency domain.
Vibration Isolation Using Mechanical Damping

- The sensor reading may be corrupted by external vibrations (i.e. from a spinning propeller).
- It may be possible to mechanically damp the vibrations.
- The base-excitation model (Fig. 4) is used to analyze vibration damping.
- Suppose that $y = \Delta \cos(\omega t)$ where $\Delta$ is the amplitude of the base excitation.
- The magnitude of the frequency response function (FRF) is given by

$$\left| \frac{X}{\Delta} \right| = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

(1)

where

$$r = \frac{\omega}{\omega_n}, \quad \omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{km}}.$$
Finding the Isolation Region

- Fig. 5 shows the magnitude plot of the FRF. Note that the base excitation is attenuated for \( r \geq \sqrt{2} \).
- The isolation region is defined as \( r \geq \sqrt{2} \).
- Tune \( k, c \) and \( m \) such that the undesirable frequencies are in the isolation region.
- For example, the natural frequency of the system can be reduced by decreasing \( k \) or increasing \( m \).
- Main point: When it comes to vibration isolation, it is not always the case that something is better than nothing.
  - It is possible to amplify the base excitation!

**Figure 5:** Magnitude plot of the frequency response function. A vertical line is drawn at \( r = \sqrt{2} \).
Stochastic Sensor Models
A typical stochastic scalar sensor model is

\[ y_{\text{meas}} = y + b + \omega \]  

(2)

where \( y \) is the true sensor value, \( b \) is a bias term and \( \omega \) is measurement noise.

The measurement noise, \( \omega \), is modeled as a zero mean, Gaussian random variable with variance \( \sigma^2 \).

The analytical autocorrelation function for this type of process is

\[ R_{xx} = \sigma^2 \delta(\tau) \]  

(3)

where \( \delta(\cdot) \) is the Dirac delta function and \( \tau \) is a lag time.

The autocorrelation function for the measurement noise is shown in Fig. 6.
Modeling the Random Bias

- The bias, $b$, is modeled as exponentially correlated noise.
- The analytical autocorrelation function for this type of process is
  \[
  R_{xx} = \sigma_b^2 \exp\left(-\frac{|\tau|}{\tau_b}\right) \quad (4)
  \]
  where $\tau_b$ is a time constant and $\sigma_b^2$ is the variance.
- Fig. 7 plots the autocorrelation function for various time constants.
- Note that the curves pass through the point \( (\tau_b, \frac{\sigma_b^2}{e}) \).

**Figure 7:** Autocorrelation function for exponentially correlated noise with $\sigma_b^2 = 0.2$ and $\tau_b = \{200, 400, 600\}$. A dashed line is drawn through the ordinate $\sigma_b^2/e$. 
Q: How do you simulate exponentially correlated noise? A: You need to determine the difference equation [1].

Let \( \{w_k\} \) be a sequence of uncorrelated, zero mean, unit variance Gaussian random variables.

The general difference equation model is

\[
x_k = \Phi x_{k-1} + G w_{k-1},
\]

where \( \Phi \) and \( G \) are unknown parameters.

To determine \( \Phi \), we multiply both sides of (5) by \( x_{k-1} \) and take the expected value:

\[
\Phi = e^{-1/\tau}.
\]

To determine \( G \), we square (5) and take the expected value:

\[
G = \sigma_b \sqrt{1 - e^{-2\tau}}.
\]

The complete model for exponentially correlated noise is then

\[
b_k = e^{-1/\tau} b_{k-1} + \sigma_b \sqrt{1 - e^{-2/\tau}} w_{k-1}.
\]
function rv = get_signal(n, tau, std1, std2, dt)
% Generates a vector of exponentially
% correlated noise plus wide-band noise.
% Inputs
%  n   = number of data points to generate
%  tau = time constant (sec)
%  std1 = standard deviation of driving noise
%  std2 = standard deviation of wide-band noise
%  dt  = sampling period (sec)
% Output
%  rv  = bias + measurement noise

f = exp(-dt / tau);
rv = zeros(n, 1);
rv(1) = std1 * randn(1);

for k = 2 : length(rv)
    rv(k) = f * rv(k-1) + std1 * sqrt(1 - f^2) * randn(1);
end

% Add measurement noise to the bias
rv = rv + std2 * randn(n, 1);
end
Bias Characterization using Allan Deviation Plots
David W. Allan developed the “Allan Variance” method to analyze the stability of oscillators.

The Allan variance can also be used to analyze the bias stability of inertial sensors.

The Allan variance chart represents the root mean square (RMS) random-drift error as a function of averaging time.

On the Allan variance chart, different random processes appear as lines with different slopes.

By plotting the Allan variance of a sensor output, one can identify the various error mechanisms at play.

The Allan variance is related to power spectral density (PSD) by [2]

$$\sigma_A^2(\tau) = 4 \int_0^\infty S_{xx}(f) \frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2} \, df. \quad (6)$$

The PSD is related to the autocorrelation function by

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f \tau} \, d\tau. \quad (7)$$
Numerical Allan Variance

- The record is divided into “bins” of a certain width.
- The bin width is proportional to an averaging time, $\tau$.
- The bin average, $\bar{y}_i$, is computed for the $i$th bin.
- The Allan Variance, $\sigma_A^2$, is the variance over all the bins of length $\tau$:
  \[
  \sigma_A^2(\tau) = \frac{1}{2} \mathbb{E} \left\langle (\bar{y}_{i+1} - \bar{y}_i)^2 \right\rangle.
  \]
- Plot $\sigma_A(\tau)$ versus $\tau$ on a log-log plot to obtain the Root Allan Variance Plot.

Figure 8: Bin averages for different averaging times, $\tau$. 
function [sig, tau] = allan(x, fs)
% Computes the Allan Deviation for the given record.
% The bin widths are given by M=2^j where j = 0,1,2,3,...
% Inputs
%   x   = Data array [DPS]
%   fs  = Sampling frequency [Hz]
% Outputs
%   sig = Allan Deviation [DPS]
%   tau = Averaging time [s]

tau0 = 1 / fs;
N = length(x);
max_clusternumber = floor(log2((N-1)/3));
tau = zeros(1, max_clusternumber + 1);
sig = zeros(1, max_clusternumber + 1);

for j = 0 : max_clusternumber
    M = 2^j; % cluster-size being evaluated
    tau(j+1) = M * tau0; % cluster time
    ii = 0;
    d = zeros(1, floor(N/M)); % Index difference vector
    for i = 1 : M : N - M + 1
        ii = ii + 1;
        d(ii) = sum(x(i:i+M-1))/M;
    end
    sig(j+1) = sqrt(0.5*mean((diff(d(1:length(d))).^2)));
end
end
Allan Deviation for White Noise

- Using (3), (7) and (6), the Allan Variance for white noise is derived to be

\[ \sigma_A^2(\tau) = \frac{\sigma^2}{\tau f_s}, \]  

(8)

where \( f_s \) is the sampling frequency.

- Taking the logarithm of (8) produces

\[ \log \sigma_A = \log \sigma - \frac{1}{2} \log \tau - \frac{1}{2} \log f_s \]

- On an Allan Deviation plot, white noise appears as a line with a slope of \(-\frac{1}{2}\).

- The standard deviation of the white noise process, \( \sigma \), is equal to the Allan deviation at \( \tau = 1/f_s \).

Figure 9: Allan Deviation of white noise with a variance of 1 and a sampling frequency of 1 Hz.
Allan Deviation for Exponentially Correlated Noise

• Using (4), (7) and (6), the Allan Variance for exponentially correlated noise is derived to be

\[
\sigma_A^2(\tau) = \frac{2\sigma_b^2\tau_b}{\tau} \left\{ 1 - \frac{\tau_b}{2\tau} \left[ 3 - 4\exp\left(-\frac{\tau}{\tau_b}\right) + \exp\left(-\frac{2\tau}{\tau_b}\right) \right] \right\}
\]  

(9)

• An approximate expression for (9) can be obtained by taking a 3\textsuperscript{rd} order Taylor Series expansion of the exponential terms:

\[
\sigma_A^2(\tau) \approx \frac{2}{3} \frac{\sigma_b^2\tau}{\tau_b}
\]  

(10)

• Taking the logarithm of (10) produces

\[
\log \sigma_A = \log \sigma_b + \frac{1}{2} \log \tau - \frac{1}{2} \log \tau_b + \frac{1}{2} \log \frac{2}{3}
\]

• On an Allan Deviation plot, exponentially correlated noise appears as a line with a slope of \(\frac{1}{2}\).
**Figure 10:** Allan deviation of exponentially correlated noise with a variance of 1 and a time constant of 500 s.
• The total Allan Variance for the sensor model is simply the sum of the Allan Variances for each individual process:

\[ \sigma_{A,\text{tot}}^2 = \sigma_{A,1}^2 + \sigma_{A,2}^2 + \ldots \]

• Assuming that the measurement is corrupted by white noise and exponentially correlated noise, the Allan Variance is

\[ \sigma_{A,\text{tot}}^2 = \frac{\sigma^2}{\tau f_s} + \frac{2}{3} \frac{\sigma_b^2 \tau}{\tau_b}. \quad (11) \]

• The Allan Deviation is given by the root of (11)

\[ \sigma_{A,\text{tot}} = \sqrt{\frac{\sigma^2}{\tau f_s} + \frac{2}{3} \frac{\sigma_b^2 \tau}{\tau_b}}. \quad (12) \]
Sensor Calibration for Tri-Axial Field Sensors
The error model for a tri-axial field sensor is given by

\[
\begin{pmatrix}
    v_1 \\
    v_2 \\
    v_3
\end{pmatrix} =
\begin{pmatrix}
    k_{11} & k_{12} & k_{13} \\
    k_{21} & k_{22} & k_{23} \\
    k_{31} & k_{32} & k_{33}
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{pmatrix} +
\begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{pmatrix}
\]

- \( \vec{u} = (u_1 \ u_2 \ u_3)^T \) is the input of the sensor.
- \( \vec{v} = (v_1 \ v_2 \ v_3)^T \) is the output of the sensor.
- \( \vec{b} = (b_1 \ b_2 \ b_3)^T \) includes the sensor bias and hard iron interferences.
- \( K \equiv (k_{ij})_{3 \times 3} \) includes all the errors caused by scale factors, misalignments and soft iron interferences.
The error included in $K$ can be classified into the following types:

- **Scale factor errors**
  
  $$C_s = \begin{pmatrix} 1 + s_x & 0 & 0 \\ 0 & 1 + s_y & 0 \\ 0 & 0 & 1 + s_z \end{pmatrix}$$

- **Misalignment between the sensor and body axes**
  
  $$C_\eta = \begin{pmatrix} 1 & -\eta_z & \eta_y \\ \eta_z & 1 & -\eta_x \\ -\eta_y & \eta_x & 1 \end{pmatrix}$$

- **Nonorthogonality and soft iron errors**
  
  $$C_\alpha = \begin{pmatrix} 1 + \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & 1 + \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & 1 + \alpha_{zz} \end{pmatrix}$$

- The matrix $K$ is the combination of the above errors, i.e. $K = C_s C_\eta C_\alpha$.

- It is unnecessary to distinguish the error sources, as they have mathematically equivalent impacts on the sensor outputs.
• The error model can be compactly written as

\[ \tilde{v} = K \tilde{u} + \tilde{b}. \]

• Solving for \( \tilde{u} \), we get

\[ \tilde{u} = L \tilde{v} - \tilde{d} \]  \hspace{1cm} (13)

where \( L \equiv K^{-1} \) and \( \tilde{d} \equiv K^{-1} \tilde{b} \)

• Sensor calibration is equivalent to determining \( L \) and \( \tilde{d} \).

• Once \( L \) and \( \tilde{d} \) have been acquired, \( \tilde{u} \) can be recovered from \( \tilde{v} \) using (13).
• The Dot Product Invariance (DPI) Method exploits the property that two vectors that are both constant in some reference frame will have a constant dot product [4].

• The DPI Method relies on some constant vector \( \vec{w} \).

• Given \( \vec{w} \), the dot product \( \vec{w} \cdot \vec{u} \) is

\[
\vec{w} \cdot \vec{u} = \vec{w}^T L \vec{v} - \vec{w}^T \vec{d} = \text{const.}
\] (14)

• This equation is linear-in-the-parameters.

• All 12 elements in \( L \) and \( \vec{d} \) may be solved using the classical LS method.

• \( \vec{w} \) can not be parallel or perpendicular to the field vector.
Physical Models for Common MEMS Sensors
MEMs Accelerometer

- Axes aligned with the aircraft body frame.
- Mounted near the aircraft CM.
- Units measured in g’s.

Figure 12: Close-up of a MEMs accelerometer.
Physical Model for a MEMs Accelerometer

- A force balance analysis of the proof mass yields the equation
  \[ m\ddot{x} = -k(x - y). \]  
  (15)
- The acceleration of the proof mass is proportional to the deflection of the suspension.
- Taking the Laplace transform of (15) gives
  \[ \frac{X(s)}{Y(s)} = \frac{1}{\frac{m}{k}s^2 + 1}. \]
- We note that
  \[ \frac{X(s)}{Y(s)} = \frac{s^2 X(s)}{s^2 Y(s)} = \frac{A_X(s)}{A_Y(s)}. \]

Figure 13: Conceptual model of an accelerometer.
The magnitude of the frequency response function (FRF) is given by

\[ |G(r)| = \left| \frac{A_X(s)}{A_Y(s)} \right| = \left| \frac{1}{1 - r^2} \right| \]

where

\[ r \equiv \frac{\omega}{\omega_n}, \quad \omega_n \equiv \sqrt{\frac{k}{m}}. \]

|G(r)| \approx 1 when \( r \ll 1 \). That is, the acceleration of the proof mass is equal to the acceleration of the housing.

**Figure 14:** Frequency response function for the accelerometer transfer function.
What Accelerometers Actually Measure

- The accelerometer measures the total acceleration minus gravity, i.e.
  \[ \vec{a} = \vec{f}/m - \vec{g}. \]  

- Consider when the input axis of the accelerometer is pointing up (Fig. 15). By noting the deflection of the proof mass, we conclude the accelerometer is registering 1 g of forward acceleration.

- In this example, the total acceleration is zero and the gravitational component along the input axis is minus 1 g. Using (16), the accelerometer output is
  \[ 0 - (-1) = 1. \]
  This agrees with our physical intuition.

*Figure 15: Accelerometer when the input axis is pointing up.*
Physical Model for a MEMs Gyroscope

- Vibrating proof mass \( m \) moves with velocity \( \vec{v} \).
- When the angular velocity \( \tilde{\Omega} \) is applied, the proof mass will experience the Coriolis acceleration \( \tilde{a}_C = 2\tilde{\Omega} \times \vec{v} \).
- The resulting displacement of proof mass is detected through a capacitive sensing structure.

**Figure 16:** (a) Close-up of a MEMs gyroscope. (b) Conceptual model of an gyroscope.
Magnetometers: Lorenz Force

- Consider a particle of charge $q$ in an electric field $\vec{E}$ (Fig. 17a). The charge will experience an electric force given by

$$\vec{F}_e = q\vec{E}.$$

- Consider a positive charge $q$ moving in a magnetic field $\vec{B}$ (Fig. 17b). The charge will experience a magnetic force given by

$$\vec{F}_m = q\vec{v} \times \vec{B}.$$

- The direction of the magnetic force is given by the right-hand rule.

- The force exerted on a charged particle in an electromagnetic field is given the the Lorenz force:

$$\vec{F} = \vec{F}_e + \vec{F}_m = q(\vec{E} + \vec{v} \times \vec{B}).$$

Figure 17: (a) Electric force exerted on a charged particle. (b) Magnetic force exerted on a moving, positive charge. The force is opposite for a negative charge moving in the same direction.
Magnetometers: Hall Effect

- A consequence of the Lorenz force is the *Hall effect*. The Hall effect is illustrated in Fig. 18.

- In this figure, current is moving through a flat conductor in the presence of a magnetic field, resulting in a magnetic force, $\vec{F}_m$.

- In order to balance the magnetic force, charge will build up on the sides of the conductor, resulting in an equal and opposite electric force, $\vec{F}_e$. This charge build-up results in a measurable voltage.

- Digital compasses make use of the Hall effect to measure the earth’s magnetic field.

*Figure 18:* The transverse voltage across the conductor is called the Hall effect. The physical principal behind the Hall effect is the Lorenz force.
• In addition to typical instrument errors, magnetometers are subject to artificial errors.

• Artificial errors are classified as hard irons (Fig. 19) or soft irons (Fig. 20).

• Hard iron denotes an unwanted magnetic source (e.g. permanent magnet). A hard iron produces the same magnetic field no matter its orientation.

• “Soft iron” denotes a magnetic distortion that depends on the incidence angle of the Earth’s magnetic field (e.g. steel).

**Figure 19:** Graphical depiction of a hard iron.  
**Figure 20:** Graphical depiction of a soft iron.
Magnetometers: Calibration

- Calibrating a magnetometer usually is done by rotating the magnetometer in a perturbation-free environment, such as a park or an open field.
- For a well-calibrated magnetometer, the sensor’s response surface will describe a sphere centered at the origin. The radius of the sphere will equal the magnitude of the earth’s magnetic field.
- An uncalibrated magnetometer will describe an off-center ellipsoid.
- The magnetometer may be calibrated using the DPI method described earlier.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 & X & Y & Z \\
\hline
X & 1 & 0 & 0 \\
Y & 0 & 1 & 0 \\
Z & 0 & 0 & 1 \\
\end{array}
\]

**Figure 21:** Locus of magnetometer measurements, before calibration (a) and after calibration (b).
Key Takeaways

- Magnetometers, accelerometers and gyroscopes form a basic sensor suite.
- The sensors are not redundant; each sensor contributes unique information.
- The attitude is resolved by fusing the various sensors.
- Each sensor is particularly susceptible to certain disturbances.
- Typically, some “fix” is applied to the sensor measurements before fusion.

**Table 1: Sensor Cheat Sheet**

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Strength</th>
<th>Weakness</th>
<th>Fix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gyro.</td>
<td>Short-term attitude estimation</td>
<td>Bias drift</td>
<td>Bias compensation</td>
</tr>
<tr>
<td>Accel.</td>
<td>Roll/Pitch estimation</td>
<td>Vibrations, inertial accelerations</td>
<td>Vibration Isolation, Acceleration compensation</td>
</tr>
<tr>
<td>Mag.</td>
<td>Yaw estimation</td>
<td>Hard irons, soft irons</td>
<td>Sensor calibration</td>
</tr>
</tbody>
</table>

